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Ordering dynamics with two non-excluding options: bilingualism in language competition

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Abstract. We consider an extension of the voter model in which a set of interacting elements (agents) can be in either of two equivalent states (A or B) or in a third additional mixed (AB) state. The model is motivated by studies of language competition dynamics, where the AB state is associated with bilingualism. We study the ordering process and associated interface and coarsening dynamics in regular lattices and small world networks. Agents in the AB state define the interfaces, changing the interfacial noise driven coarsening of the voter model to curvature driven coarsening. This change in the coarsening mechanism is also shown to originate for a class of perturbations of the voter model dynamics. When interaction is through a small world network the AB agents restore coarsening, eliminating the metastable states of the voter model. The characteristic time to reach the absorbing state scales with system size as $\tau \sim \ln N$ to be compared with the result $\tau \sim N$ for the voter model in a small world network.

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Contents

1. Introduction

Understanding the complex collective behaviour of many-particle systems in terms of a microscopic description based on the interaction rules among the particles is the well-established purpose of statistical physics. This micro-macro paradigm [1] is also shared by social science studies based on agent interactions (agent based models). In many cases, parallel research in both disciplines goes far beyond superficial analogies. For example, Schelling's model [1] of residential segregation is mathematically equivalent to the zero-temperature spin-exchange kinetic Ising model with vacancies. Cross-fertilization between these research fields opens interesting new topics of research [2]. In this context the consensus problem is a general one of broad interest: the question is to establish when the dynamics of a set of interacting agents that can choose among several options leads to a consensus in one of these options, or alternatively, when a state with several coexisting social options prevails [3]. For equilibrium system the analogy would be with an order-disorder transition. For non-equilibrium dynamics we rely on ideas of studies of domain growth and coarsening in the kinetics of phase transitions [4], where dynamics is dominated by interface motion. Microscopic interaction rules include two ingredients that determine the ultimate fate of the system, either homogenous consensus state or spatial coexistence of domains of different options. These ingredients are: (i) the interaction mechanism between particles/agents, and (ii) the network of interactions. Interactions in complex networks is a relatively recent paradigm in statistical physics [5]. A general still open question is the study of coarsening in complex networks [6].

Language competition is a particular example of consensus problems that motivates the present study. It refers to the dynamics of language use in a multilingual social system due to individuals interacting in a social network. Recent interest in this problem has been triggered by the model proposed by Abrams and Strogatz (AS-model) [7] to account for data of extinction of endangered languages [8]. Other different problems of language dynamics include those of language evolution (dynamics of language structure) and language cognition (learning processes). Among these, *semiotic dynamics*, considered in the context of the *naming game* [9], is also an example of consensus problems. The seminal paper of Abrams and Strogatz [7], as well as others along the same lines [10]–[13], belong to the general class of mean-field population dynamics studies based on nonlinear ordinary differential equations for the populations of speakers of different languages. Other studies implement microscopic agent-based-models with speakers of many or few languages [13]–[16] as reviewed in [17].

The microscopic version [16] of the AS-model for the competition of two equivalent languages is equivalent to the voter model [18]–[23]. The voter model is a prototype lattice spin-model of non-equilibrium dynamics for which $d = 2$ is a critical dimension [20]: for regular lattices with $d > 2$ coarsening does not occur and, in the thermodynamic limit, the system does not reach one of the homogenous absorbing states (consensus states). The same phenomenon occurs in complex networks of interaction of effective large dimensionality where a finite system gets trapped in long-lived heterogeneous metastable states [21]–[23]. From the point of view of interaction mechanisms, the voter model is one of random imitation of a state of a neighbour. A different mechanism (for $d > 1$) of majority rule is the one implemented in a zero-temperature spin-flip kinetic Ising (SFKI) model.[2] Detailed comparative studies of the consequences of these two mechanisms in different interaction networks have been recently reported [26]. From the point of view of coarsening and interface dynamics, a main difference is that, in the voter model coarsening is driven by interfacial noise, while for a SFKI model coarsening is curvature driven with surface tension reduction.

The voter and SFKI models are two-option models (spin +1 and spin −1) with two equivalent global attractors for the system. Kinetics of multi-option models like Potts or clock models were addressed long ago [27]. More recently, a related model proposed by Axelrod [28] has been studied in some detail [29, 30]. This is a multi-option model but, in general, its non-equilibrium dynamics does not minimize a potential leading to a thermodynamic equilibrium state like in traditional statistical physics [31]. On the other hand, the kinetics of the simplest three-options models [32]–[34] has not been studied in great detail.

We are here interested in the class of three-state models for which two states are equivalent (spin ±1, state A or B) and a third one is not (spin 0, state AB). Different dynamical microscopic rules can be implemented for such choice of individual states, some of which can be regarded as constrained voter-model dynamics [33]. The choice of dynamical rules in this paper is dictated by our motivation of considering bilingual individuals in the competition dynamics of two languages [12, 13]. We will consider here two socially equivalent languages. The possible state of the agents are speaking either of these languages (A or B) or a third non-equivalent bilingual state (AB). In the context of the consensus problem this introduces a special ingredient in the sense that the options are not excluding: there is a possible state of the agents (bilinguals) in which there is coexistence of two possible options. In a more general framework, the problem addressed here is that of competition or emergence of social norms [35] in the case where two norms can coexist at the individual level.

In this paper, and building upon a proposal by Minett and Wang [13], we study a microscopic model of language competition which reduces to the microscopic AS-model [16] when bilingual agents are not taken into account. Our presentation in the remaining sections of the paper is of general nature for the abstract problem of the ordering dynamics of an extension of the voter model in which a third mixed AB state is allowed. We aim to explore possible mechanisms for the stabilization of two options' coexistence, possible metastable sates, and the role of AB states (bilingual individuals) and interaction network (social structure) in these processes. To this end, we analyse the growth mechanisms of A or B spatial domains (monolingual domains), the dynamics at the interfaces (linguistic borders), and the role of AB states (bilingual individuals) in processes of domain growth. This is done in regular lattices and in complex networks of interaction. Generally speaking, we find that allowing for the AB state (bilinguals)

[2] A different majority rule based in group interaction is considered in [24, 25].

modifies the nature and dynamics of interfaces: agents in the AB state define thin interfaces and coarsening processes change from voter-like dynamics to curvature driven dynamics. This change of coarsening mechanism is also shown to originate for a class of perturbations of the voter model.

The outline of the paper is as follows: section 2 describes our microscopic model which is analysed in a two-dimensional (2D) regular lattice in section 3. In section 4, we describe the dynamics of the model in a small world network [36]. Section 5 contains a summary of our results.

2. A model with two non-excluding options

We consider a model in which an agent i sits in a node within a network of N individuals and has k_i neighbours. It can be in three possible states: A, agent choosing option A (using language A: monolingual A); B, agent choosing option B (using language B: monolingual B); and AB, agent in a state of coexisting options (bilingual agent using both languages, A and B).

The state of an agent evolves according to the following rules: starting from a given initial condition, at each iteration we choose one agent i at random and we compute the local densities for each of the three communities in the neighbourhood of node i, σ_l ($l = A, B, AB$). The agent changes its state according to the following transition probabilities

$$p_{A\to AB} = \tfrac{1}{2}\sigma_B, \qquad p_{B\to AB} = \tfrac{1}{2}\sigma_A; \tag{1}$$

$$p_{AB\to B} = \tfrac{1}{2}(1-\sigma_A), \qquad p_{AB\to A} = \tfrac{1}{2}(1-\sigma_B). \tag{2}$$

Equation (1) gives the probabilities for an agent i to move away from a single-option community, A or B, to the AB community. They are proportional to the density of agents in the opposed single-option state in the neighbourhood of i. On the other hand, equation (2) gives the probabilities for an agent to move from the AB community towards the A or B communities. They are proportional to the local density of agents with the option to be adopted, including those in the AB state ($1-\sigma_l = \sigma_j + \sigma_{AB}, l, j = A, B; l \neq j$). It is important to note that a change from state A to state B or vice versa, always implies an intermediate step through the AB state. The dynamical rules (1) and (2) are fully symmetric under the exchange of A and B, so that states A and B are equivalent with no preference for any of the two options. Reaching consensus in either of these two states is a symmetry breaking process.[3]

These dynamical rules, which define a modification of the two state voter model to account for a third mixed AB state, reflect the special character of this state as one of coexisting options. In the remainder of this paper, we will refer to the model defined by (1) and (2) as the AB-model. For the voter model the transition probabilities are simply given by $p_{A\to B} = \sigma_B$, $p_{B\to A} = \sigma_A$. The voter model rules are equivalent to the adoption by the agents of the opinion of a randomly chosen neighbour.

[3] Non-equivalent options were considered in the original version of the model [13]. The prefactor $1/2$ in (1) and (2) corresponds to the special case of equivalence between A and B. Here it can be interpreted as an inertia factor that sets a maximum probability of $1/2$ for changing the state, introducing a microscopic time scale.

In a fully connected network and in the limit of infinite population size, the AB-model can be described by coupled differential equations for the total population densities $(\Sigma_A, \Sigma_B, \Sigma_{AB})$ [13]

$$\mathrm{d}\Sigma_A/\mathrm{d}t = 1/2[1 - \Sigma_A + (\Sigma_B)^2 - 2\Sigma_B]; \tag{3}$$

$$\mathrm{d}\Sigma_B/\mathrm{d}t = 1/2[1 - \Sigma_B + (\Sigma_A)^2 - 2\Sigma_A]. \tag{4}$$

The analysis of these mean field equations shows the existence of three fixed points: two of them stable and equivalent, corresponding to consensus in the state A or B: $(\Sigma_{\mathrm{A}}, \Sigma_{\mathrm{B}}, \Sigma_{\mathrm{AB}}) = (1, 0, 0), (0, 1, 0)$; and another one unstable, with non-vanishing values for the global densities of agents in the three states: $(\Sigma^*_A, \Sigma^*_B, \Sigma^*_{AB})$, with $\Sigma^*_l \neq 0\,(l = A, B, AB)$. On the other hand, in the same approximation, the voter model reduces to the equation:

$$\mathrm{d}\Sigma_A/\mathrm{d}t = 0, \tag{5}$$

predicting that any given initial density of agents in state A would persist forever. Simulations of the corresponding stochastic discrete voter model in a lattice [16, 20] indicate a very different behaviour with one of the two options eventually becoming dominant. In this paper, we go beyond the simple mean field description (3) and (4) describing the microscopic dynamics in which discrete and finite size effects, as well as the topology of the network of interaction are taken into account.

In our simulations, we use random asynchronous node update: at each iteration or time step a single node is randomly chosen and updated according to the transiton probabilities (1) and (2). We normalize time so that in every unit of time each node has been updated on average once. Therefore, a unit of time includes N time steps. In all our simulations we start from random initial conditions: random distribution in the network of $1/3$ of the population in state A, $1/3$ in state B and $1/3$ in state AB.

For a quantitative description of the ordering dynamics towards consensus in the A or B state we use as an order parameter the ensemble average interface density $\langle\rho\rangle$. This is defined as the density of links joining nodes in the network which are in different states [20, 22]. The ensemble average, indicated as $\langle..\rangle$, denotes average over realizations of the stochastic dynamics starting from different random initial conditions. For our random initial conditions $\langle\rho(t = 0)\rangle = 2/3$: a given node has probability $2/3$ of being connected to a node in a different state. This is valid for any network in which there are no correlations in the random initial distribution of states among the nodes of the network. During the time evolution, the decrease of ρ from its initial value describes the ordering dynamics by a coarsening process with growth of spatial domains in which agents are in the same state. The minimum value $\rho = 0$ corresponds to an absorbing state where all the agents have reached consensus in the same state.

3. Coarsening in a regular lattice

We first consider the dynamics of the AB-model defined in the previous section on a 2D regular lattice with four neighbours per node. In figure 1 we show, for a typical realization, the time evolution of the total densities of agents in state A, Σ_A, in state B, Σ_B, and in state AB, Σ_{AB}; and the density of interfaces, ρ. State A takes over the system, while the opposite option B disappears. Consensus in either of the two equivalent states A or B is always reached

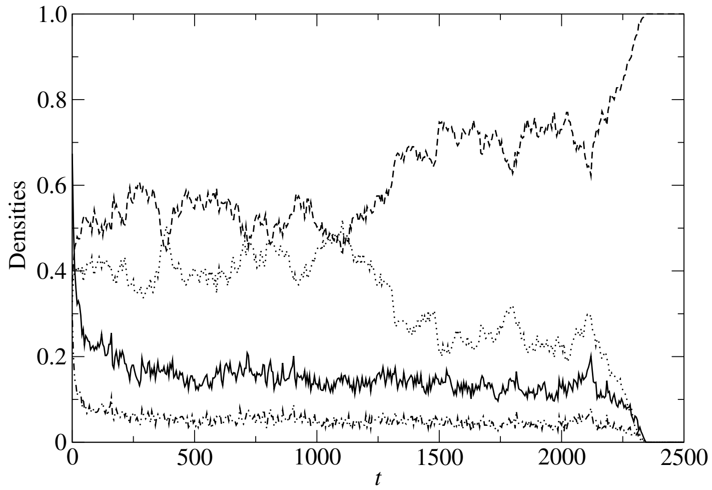

Figure 1. Time evolution of the total densities of agents in the three states, Σ_i ($i = A, B, AB$), and the interface density, ρ for the AB-model in a 2D regular lattice. One realization in a population of $N = 400$ agents is shown. From top to bottom: Σ_A (dashed line), Σ_B (dotted line), ρ (solid line), Σ_{AB} (dot-dashed line).

(with equal probability to reach consensus in state A or B). We observe an early very fast decay of the interface density and of the total density of agents in the state AB, Σ_{AB}, followed by a slower decay corresponding to the coarsening dynamical stage. This stage lasts until a finite size fluctuation causes the dominance of one of the states A or B, and the density of AB agents disappears together with the density of agents in the state opposite to the one that becomes dominant.

In figure 2 we show the time evolution of the average interface density and of the average total density of AB agents, averaged over different realizations. For the relaxation towards one of the absorbing states (dominance of either A or B) both the average interface density and the average density of AB agents decay following a power law with the same exponent, $\langle\rho\rangle \sim \langle\Sigma_{AB}\rangle \sim t^{-\gamma}$, $\gamma \simeq 0.45$. This indicates that the evolution of the average density of the AB agents is correlated with the interface dynamics. Several systems sizes are shown in order to see the effect of finite size fluctuations. During the coarsening stage described by the power law behaviour, spatial domains of the A and B community are formed and grow in size. From the dependence of $\langle\rho\rangle$ with time, it follows that the typical size of a domain, $\langle\xi\rangle$, grows as $\langle\xi\rangle \sim t^{0.45}$. Eventually a finite size fluctuation occurs (as the one shown in figure 1) so that the whole system is taken to an absorbing state in which there is consensus in either the A or B option.

During the coarsening process spatial domains of AB agents are never formed. Rather, during an early fast dynamics AB agents place themselves in the boundaries between A and B domains (movie 1b in supplementary material). This explains the finding that the density of AB agents follows the same power law than the average density of interfaces. We have also checked the intrinsic instability of an AB community: an initial AB domain disintegrates very fast into smaller A and B domains, with AB agents just placed at the interfaces.

Our result for the growth law of the characteristic length of a A or B domain is compatible with the well known exponent 0.5 associated with domain growth driven by mean curvature and surface tension reduction observed in SFKI models [4]. However, systematic deviations from the exponent 0.5 are observed. These deviations are at least partially due to the fact that on

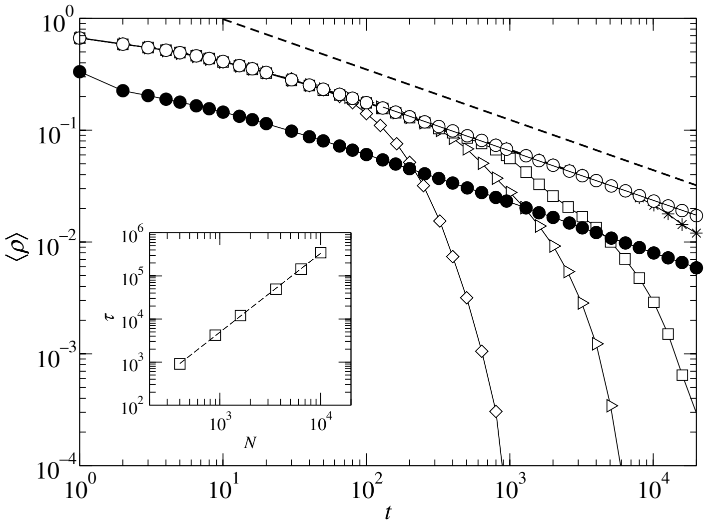

Figure 2. Time evolution of the averaged interface density $\langle\rho\rangle$ for the AB-model in a 2D regular lattice for different system sizes. Empty symbols: from left to right: $N = 10^2$ (◇), 20^2 (▹), 30^2 (□), 100^2 ($*$), 300^2 (○). The averaged global density of AB agents, $\langle\Sigma_{AB}\rangle$, for $N = 300^2$ agents is also shown (●). Averages are calculated over 100–1000 realizations depending on the system size. Dashed line for reference: $\langle\rho\rangle \sim t^{-0.45}$. Inset: dependence of the characteristic time to reach an absorbing state τ with the system size: $\tau \sim N^{1.8}$ (computed from numerical analysis of the distribution of survival times for the dynamical metastable states of figures 3 and 4-left).

closer inspection there are two type of qualitatively different realizations, which we show in figure 3: while many of them have a coarsening stage followed by a finite size fluctuation which drives the system to an absorbing state, a finite fraction of the realizations (around 1/3 of them, depending on system sizes) get trapped in long-lived metastable states. These metastable states are reminiscent of the ones found [37] in the analysis of a two states majority rule dynamics based on group interaction [24]. They correspond to stripe-like configurations for an A or B domain. The boundaries of these stripe-shaped domains are close to flat interfaces but with interfacial noise present (figure 4-left, and movie 2a in the supplementary material). Although long-lived, these configurations continue to evolve and in this sense they are different from the stripe-like frozen states with completely flat boundaries found in a zero temperature SFKI model [38]. When a realization falls in such dynamical metastable states, coarsening stops (the average interface density fluctuates around a fixed value), until eventually a finite size fluctuation takes the system to one of the absorbing states.

If the realizations that fall into long-lived dynamical metastable states are removed when computing the averaged interface density, the power law exponent for the decay of $\langle\rho\rangle$ increases, approaching the value $\gamma = 0.5$ characteristic of curvature driven coarsening. Other deviations from the exponent $\gamma = 0.5$ can be due to non-trivial logarithmic corrections. In 3D lattices, we also find an exponent close to 0.5, which substantiates the claim that curvature reduction is the dominant mechanism at work for the coarsening process in the AB-model. The existence of two type of realizations gives rise to two different characteristic times. For the realizations that do

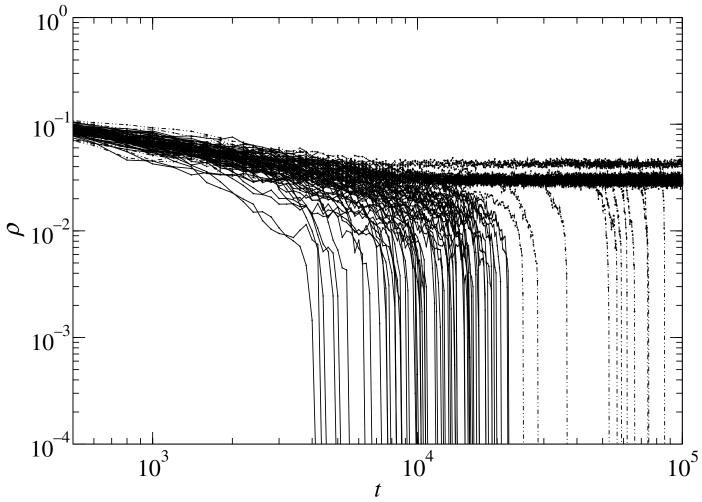

Figure 3. The time dependence of the interface density ρ in a regular lattice for the AB-model is shown for 100 realizations. We observe two types of realizations: most of them decay by a finite size fluctuation to an absorbing state after the stage of coarsening (solid lines); however, around 1/3 of them get trapped in dynamical metastable states, identified by an essentially constant value of ρ, until they eventually decay (dotted lines).

Figure 4. Snapshots of simulations which get trapped in stripe-like dynamical metastable states. $N = 50^2$ agents. Legend: red: state A, blue: state B, white: state AB. Left panel: AB-model defined in section 2. 2D regular lattice; 4 neighbours per site. Right panel: ϵ-model ($\epsilon = 1.0$). 2D regular lattice; 8 neighbours per site. (See also movie 2a and movie 2b in the supplementary material.)

not get trapped in long lived metastable states, the characteristic time to reach an absorbing state can be estimated to scale as $\tau \sim N$ since the coarsening is described by $\langle\rho\rangle \sim t^{-\gamma}$, with $\gamma \simeq 0.5$, and at the time of reaching consensus $\langle\rho\rangle \sim (1/N)^{1/d}$ (d is the dimensionality of the lattice). Focusing now in the realizations that involve a metastable state we find that their distribution of survival times, i.e., the time needed for a stripe-like configuration to reach an absorbing state, displays an exponential tail. A numerical analysis of these distributions results in a dependence on system size given by $\tau \sim N^{\alpha}$, with $\alpha \simeq 1.8$, as shown in the inset of figure 2. When taking into account all realizations, the global characteristic time to reach an absorbing state for large system sizes is dominated by the persistence of the dynamical metastable states, so that $\tau \sim N^{1.8}$.

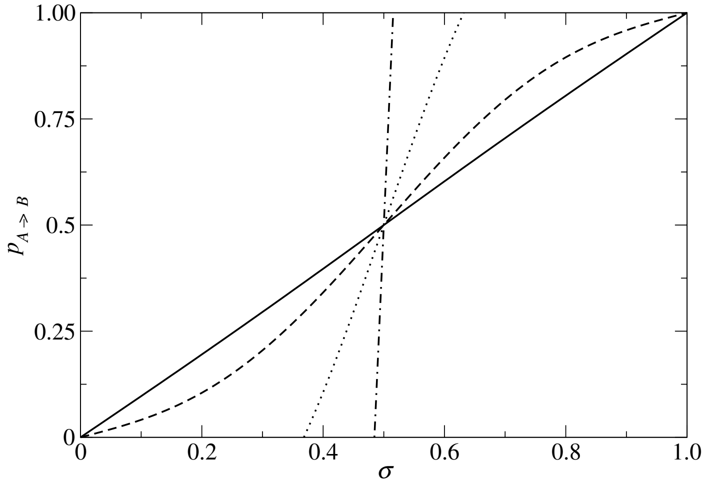

Figure 5. Transition probabilities (equation (6)) for the ϵ-model for different values of ϵ. When $\epsilon > 1/2\pi$ the transition probability for such a given ϵ is defined as follows: $p_{A\to B}$ as given by equation (6) for values of σ such that $0 \leqslant p_{A\to B} \leqslant 1$; $p_{A\to B} = 0[1]$ for values of σ such that equation (6) gives $p_{A\to B} < 0[>1]$. The limit $\epsilon \to \infty$, corresponds to the step-function transition probability of the SFKI model at zero temperature. $\epsilon = 0.01$ (solid line), 0.2 (dashed), 1.0 (dotted), 10.0 (dot-dashed).

The AB-model analysed here is a modification of the two state voter model. For the voter model coarsening in a $d = 2$ square lattice occurs by a different mechanism, interfacial noise, such that $\langle\rho\rangle \sim (\ln t)^{-1}$ [19, 20]. For a finite system the characteristic time to reach an absorbing state scales as $\tau \sim N \ln N$ [16, 39]. Therefore, the introduction of the AB state is identified as a mechanism to modify the interface dynamics from interfacial noise to curvature driven dynamics. In spite of the small number of AB agents that survive in the dynamical process, they cause a non-trivial modification of the dynamics. Indeed, in our simulations we observe the formation of well defined interfaces between A and B domains, populated by AB agents, that evolve by a curvature driven mechanism. The different nature of the coarsening process is illustrated by comparing movie 1a (voter model) with movie 1b (AB-model) in the supplementary material. On the qualitative side, the inclusion of the AB agents gives rise to a much faster coarsening process, but due to the existence of dynamical metastable states, on the average it also favours a longer dynamical transient in which domains of the two competing options coexist (larger lifetime before reaching the absorbing state for large fixed N).

A natural question that these results pose is if the crossover from interfacial noise dynamics of the voter model to curvature driven dynamics is generic for any structural modification of the voter model. In order to interpolate from the voter model dynamics towards the majority model represented by the zero-temperature SFKI model where the dynamics is curvature driven, we have considered the coarsening process in a 2D lattice in which agents can choose between two excluding options (states A and B) and the dynamic rules are as defined above but with transition probabilities (see figure 5):

$$p_{A\to B} = \sigma_B - \epsilon \sin(2\pi\sigma_B), \quad p_{B\to A} = \sigma_A - \epsilon \sin(2\pi\sigma_A), \quad \epsilon \leqslant \frac{1}{2\pi} \tag{6}$$

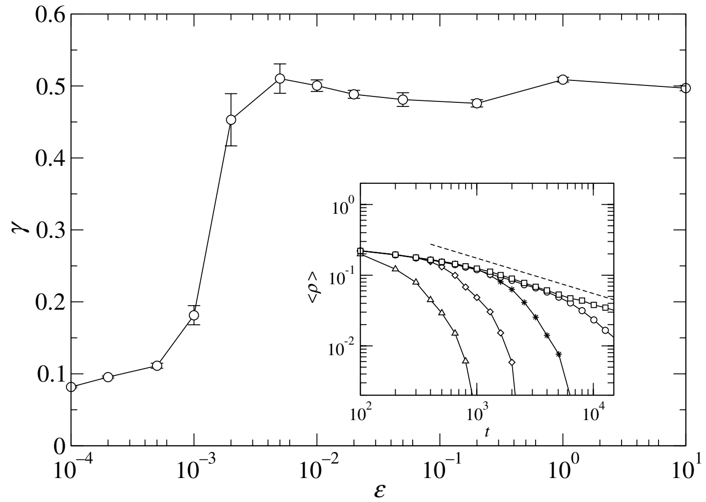

Figure 6. Characteristic coarsening exponent γ ($\langle\rho\rangle \sim t^{-\gamma}$) for the ϵ-model as a function of the perturbation parameter ϵ. $N = 400^2$ agents. Averages taken over 75 realizations. Inset: time evolution of the average interface density. From left to right: $N = 20^2$ ($\triangle$), 50^2 ($\diamond$), 100^2 ($*$), 200^2 ($\circ$), 400^2 ($\square$) agents. Averages taken over 100 realizations. Given a value of ϵ ($\epsilon = 0.01$ in this figure), a power law for the average interface density decay is found for large enough system sizes. Dashed line for reference $\langle\rho\rangle \sim t^{-0.5}$.

In the remainder of this paper, we will call this modification of the voter model the ϵ-model. The parameter ϵ measures the strength of the term that perturbs the interaction rules of the voter model ($\epsilon = 0$). This perturbation of the voter model implies that the probability of changing option is no longer a linear function of the density of neighbouring agents in the option to be adopted. With the perturbation term chosen here there is a nonlinear reinforcing (of order ϵ) of the effect of the local majority: the probability to make the change $A \to B$ is larger (smaller) than σ_{B} when $\sigma_{\mathrm{B}} > 1/2$ [$\sigma_{\mathrm{B}} < 1/2$]. In particular, we note that for $\epsilon \neq 0$, the conservation law of the ensemble average magnetization, a characteristic symmetry of the voter model, is no longer fulfilled. For later comparison we recall that in the zero-temperature SFKI model the local majority determines, with probability one, the change of option: $p_{A\to B} = 1[0]$ if $\sigma_B > 1/2$ [$\sigma_B < 1/2$].

Our results for the exponent γ in a power law fitting $\langle\rho\rangle \sim t^{-\gamma}$ for the ϵ-model are shown in figure 6 for different values of ϵ. In these simulations, we have considered a 2D lattice with eight neighbours per node so that more values are allowed for the perturbation term in equation (6). For very small values of ϵ, we observe an exponent $\gamma \simeq 0.1$ compatible with the logarithmic decay ($\langle\rho\rangle \sim (\ln t)^{-1}$) of the voter model, as obtained in [16]. However, for small, but significant values of ϵ there is a crossover to a value $\gamma \simeq 0.5$ associated with curvature driven coarsening. Dynamical metastable states analogous to the ones found in the model of section 2 are also found (figure 4-right, and movie 2b in the supplementary material) with probability $\sim 1/3$ (depending on system sizes) for values of ϵ for which $\gamma \simeq 0.5$. The $1/3$ fraction of realizations corresponds to the probability to reach a frozen configuration in the SFKI at zero temperature [38]. The distribution of survival times of these dynamical metastable states also displays an exponential tail. These realizations have been removed to calculate the value of γ.

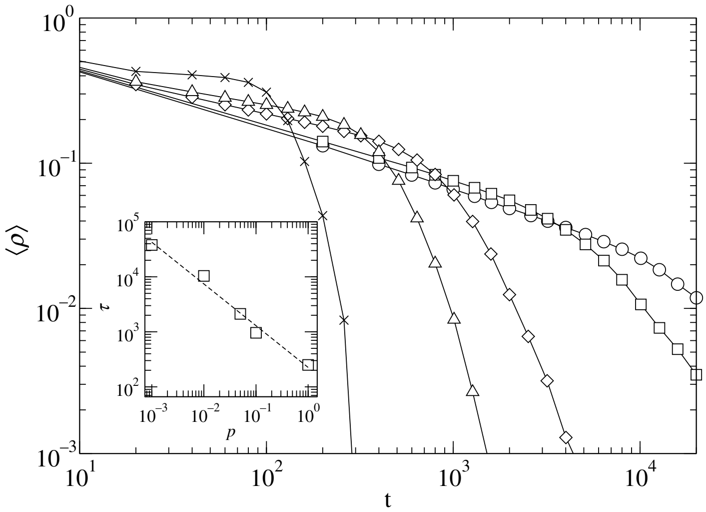

Figure 7. Time evolution of the average interface density $\langle\rho\rangle$ in small world networks with different values of the rewiring parameter p. From left to right: $p = 1.0(\times)$, 0.1 ($\triangle$), 0.05 ($\diamond$), 0.01 ($\square$), 0.0 ($\bigcirc$). For comparison the case $p = 0$ for a regular network and the case $p = 1$ corresponding to a random network are also shown. The inset shows the dependence of the characteristic time to reach an absorbing state τ with the rewiring parameter p. The dashed line corresponds to the power law fit $\tau \sim p^{-0.76}$. Population of $N = 100^2$ agents. Averages taken over 500 realizations.

We conclude that a small perturbation of the transition probabilities of the voter model dynamics such that the dependence on local density σ is no longer linear, leads to a new interface dynamics, equivalent to the one found in section 2 by including a third state where options are non-excluding. This illustrates the fact that the voter model dynamics is very sensitive to perturbations of its dynamical rules.

4. Coarsening in a small world network

We next consider the dynamics of the AB-model on a small world network constructed following the algorithm of Watts and Strogatz [36]: starting from a 2D regular lattice with four neighbours per node, we rewire with probability p each of the links at random, getting in this way a partially disordered network with long range interactions throughout it.

In figure 7 we show the evolution of the average interface density for different values of p. As we found in the regular lattice, we also observe here a dynamical stage of coarsening with a power law decrease of $\langle\rho\rangle$ followed by a fast decay to the A or B absorbing states caused by a finite size fluctuation. During the dynamical stage of coarsening, the A and B communities have similar size, while the total density of AB agents is much smaller. In the range of intermediate values of p properly corresponding to a small world network, increasing the rewiring parameter p has two main effects: (i) the coarsening process is notably slower; (ii) the characteristic time to reach an absorbing state τ, which can be computed here as the time when $\langle\rho\rangle$ sinks below a given small value, drops following a power law (inset of figure 7): $\tau \sim p^{-0.76}$, so that the absorbing state is reached much faster as the network becomes disordered.

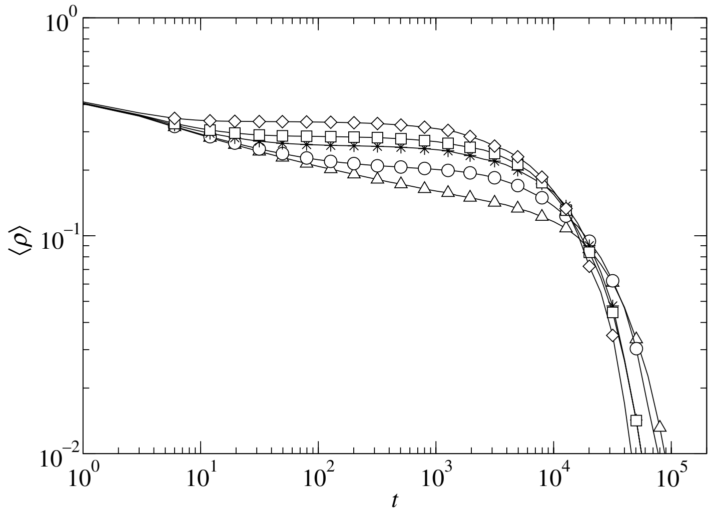

Figure 8. Time evolution of the average interface density $\langle\rho\rangle$ for the voter model in a small world network with different values of p. From up to bottom, $p = 1.0$ ($\diamond$), 0.1 ($\square$), 0.05 ($*$), 0.01 ($\circ$), 0.0 ($\triangle$). Population of $N = 100^2$ agents. Averages taken over 900 realizations.

To understand the role of the AB state in the ordering dynamics in a small world network, the results of figure 7 should be compared with the ones in figure 8 for the two state voter model in the same small world network.[4] In contrast with the AB-model, moderate values of p stop the coarsening process of a two-state voter model leading to dynamical metastable states characterized by a plateau regime for the average interface density [21, 22]. However, the lifetime of these states is not very sensitive to the value of p, with the characteristic time to reach an absorbing state being just slightly smaller than the one obtained in a regular lattice ($p = 0$). This is a different effect than the strong dependence on p found for the characteristic time to reach an absorbing state when AB agents are included in the dynamics. Comparing the results of figures 7 and 8 for a fixed intermediate value of p, we observe that including AB agents in the dynamics on a small world network of interactions allows the coarsening process to take place, and it also produces an earlier decay to the absorbing state.

System size dependence for a fixed value of the rewiring parameter p is analysed in figure 9. We observe that the initial stage of coarsening process is grossly independent of system size, but the characteristic time to reach an absorbing state scales with the system size N as $\tau \sim \ln(N)$. For the two state voter model $\tau \sim N$ [22]. Therefore, the faster decay to the absorbing state caused by the presence of AB agents in a large system interacting through a small world network is measured by the ratio $\frac{\tau_{AB}}{\tau_{\mathrm{voter}}}|_{SW} \sim \frac{\ln(N)}{N}$. We note that this faster decay is qualitatively the opposite result than the one found in a regular lattice where $\tau_{AB} \sim N^{1.8} > \tau_{\mathrm{voter}} \sim N\ln(N)$. That is, on the average the AB agents in a regular lattice for a large system slow down the decay towards the absorbing state due to the dominance of the dynamical metastable states described in section 3.

[4] Note that the small world network considered in [21] is obtained by a rewiring process of a $d = 1$ regular lattice.

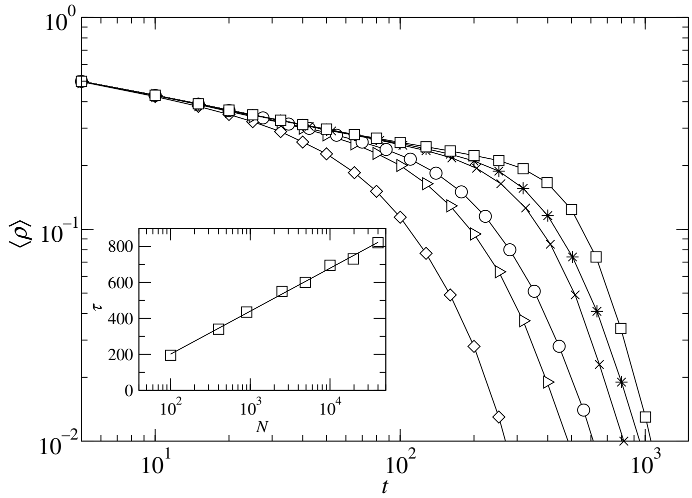

Figure 9. Time evolution of the averaged interface density, $\langle\rho\rangle$, for different values of the population size, N, in a small world network with $p = 0.1$. $N = 10^2$ ($\diamond$), 20^2 ($\triangle$), 30^2 ($\bigcirc$), 70^2 ($\times$), 100^2 ($*$), 200^2 ($\square$); from left to right. Averaged over 1000 realizations in 10 different networks. Inset: dependence of the characteristic time to reach an absorbing state τ (computed as the time when $\langle\rho\rangle$ sinks below a given small value; 0.03 in this figure) with the system size: $\tau \sim \ln N$.

5. Summary and conclusions

We have studied the non-equilibrium transient dynamics of approach to the absorbing state for the AB-model, an extension of the voter model in which the interacting agents can be in either of two equivalent states (A or B) or in a third mixed state (AB). A global consensus state (A or B) is reached with probability one. A domain of agents in the AB state is not stable and the density of AB-agents becomes very small after an initial fast transient, with AB agents placing themselves in the interfaces between single-option domains. In spite of these facts, the AB-agents produce an essential modification of the processes of coarsening and domain growth, changing the interfacial noise dynamics of the voter model into a curvature driven interface dynamics characteristic of two-option models with updating rules based on local majorities. This change in the coarsening mechanism is also found for small perturbations (ϵ-model) of the random imitation dynamics of the voter model that destroy the linear dependence of the transition probabilities on the local density. This result indicates that the effect might be generic for small structural modifications of the voter model dynamical rules. We have also shown that in a 2D regular lattice, the system reaches dynamical metastable states with a probability $\sim 1/3$ in both, the AB-model and the ϵ-model. The effect of the topology of the network of interactions has been addressed considering a small world network. While for the original two-state voter model the small world topology results in long lived metastable states in which coarsening has become to a halt [21, 22], the AB-agents restore the processes of coarsening and domain growth. Additionally, they speed-up the decay to the absorbing state by a finite size fluctuation. We obtain a characteristic time to reach an absorbing state that scales with system size as $\tau \sim \ln N$ to be compared with the result $\tau \sim N$ for the voter model.

From the point of view of recent studies of linguistic dynamics, our extension of the voter model allowing for two non-excluding options, the AB-model, is a generalization of the microscopic version [16] of the Abrams–Strogatz model [7] for two socially equivalent languages, to include the effects of bilingualism (AB-agents) [13] and social structure. Within the assumptions and limitations of our model, our results imply that bilingualism is not an efficient mechanism to stabilize language diversity when a social structure of interactions such as the small world network is taken into account. In contrast, bilingual agents are generally found to ease the approach to absorbing monolingual states by an obvious effect of smoothing the communication across linguistic borders.

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